# Junior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

## Solutions and investigations

**26 April 2023**

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to challenges@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, sometimes, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).



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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
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**1.** What is the value of $3202 - 2023$?

A 821 B 1001 C 1179 D 1221 E 1279

SOLUTION **C**

A standard subtraction sum gives

$$\begin{array}{r} 3202 \\ -\ \underline{2023} \\ \underline{1179}. \end{array}$$

Therefore the value of $3202 - 2023$ is 1179.

FOR INVESTIGATION

**1.1** (a) What is the value of $4202 - 2024$?

(b) What is the value of $5204 - 4025$?

(c) Find the highest common factor of your answers to Problems 1.1 (a) and (b).

**1.2** For each positive integer $n$ let $n^*$ be the integer obtained from $n$ by reversing the order of its digits. For example $4321^* = 1234$.

(a) Prove that for every four-digit integer $n$ the number that is the highest common factor of the answers to Problems 1.1 (a) and (b) is also a factor of the number $n - n^*$.

(b) Investigate whether your answer to Problem 1.2 (a) generalizes to cover the case of integers with other than four digits.

(c) Let $n = 4321$, and $m = n - n^*$. Check that $m + m^* = 10\,890$.

(d) In 1.2(c) you were asked to check that if $n = 4321$ and $m = n - n^*$, then $m + m^* = 10\,890$. Suppose that $n$ is some other four-digit positive integer with $n \geq n^*$. What are the other possible values of $m + m^*$?

**2.** How many of the following five options are factors of 30?

A 1 B 2 C 3 D 4 E 5

SOLUTION **D**

We have $30 = 1 \times 30 = 2 \times 15 = 3 \times 10 = 5 \times 6$. Therefore 1, 2, 3 and 5 are all factors of 30.

Since $30 = 4 \times 7.5$, and 7.5 is not an integer, 4 is not a factor of 30.

We therefore see that four of the given options are factors of 30.

FOR INVESTIGATION

**2.1** How many different positive integers are factors of 30?

**2.2** (a) Check that the number 42 has the same number of factors as 30.

(b) Find another positive integer that has the same number of factors as 30 and 42.

**3.** What is the value of $\dfrac{1+2+3+4+5}{6+7+8+9+10}$?

A $\dfrac{1}{2}$ B $\dfrac{3}{8}$ C $\dfrac{7}{16}$ D $\dfrac{9}{20}$ E $\dfrac{1}{3}$

SOLUTION **B**

METHOD 1

We have $1+2+3+4+5 = 15$ and $6+7+8+9+10 = 40$. Therefore

$$\frac{1+2+3+4+5}{6+7+8+9+10} = \frac{15}{40} = \frac{3\times 5}{5\times 8} = \frac{3}{8}.$$

METHOD 2

Method 1 is very straightforward, and is quite efficient as it only involves calculating two sums of five integers. However, it makes no use of the fact that the sums are sums of *consecutive integers*. It would involve a lot more work if many more than five integers were involved in the two sums.

In the second method we make use of a general formula for the sum of an *arithmetic sequence*, that is, a sequence of numbers with a common difference. In a sequence of consecutive integers there is a common difference of 1. Therefore it is a special case of an arithmetic sequence.

The sum of a finite sequence of numbers is equal to the number of terms in the sequence multiplied by the mean value of the terms. When we have an arithmetic sequence containing an odd number of terms, the mean value is just the middle term of the sequence. (When the number of terms is even, the mean is the mean of the first and last terms,)

It follows that when, as here, the two arithmetic sequences contain the same odd number of terms the sum of the terms of the first sequence divided by the sum of the terms of the second sequence is equal to the middle term of the first sequence divided by the middle term of the second sequence.

Using the facts explained above, we have

$$\frac{1+2+\mathbf{3}+4+5}{6+7+\mathbf{8}+9+10} = \frac{3}{8}.$$

FOR INVESTIGATION

**3.1** Find the values of the following expressions

(a) $\dfrac{1+2+3+4+5+6+7+8+9+10+11+12+13+14+15}{16+17+18+19+20+21+22+23+24+25+26+27+28+29+30}$,

(b) $\dfrac{37+40+43+46+49+52+55}{57+61+65+69+73+77+81}$.

**4.** One of these is the largest two-digit positive integer that is divisible by the product of its digits.
Which is it?

A 12 B 24 C 36 D 72 E 96

SOLUTION **C**

In the context of the JMC we are entitled to assume that one of the given options is the largest two-digit positive integer that is divisible by the product of its digits. So it is only necessary to look to see which is the largest of the given options that has the required property. So we check the options in decreasing order.

We have

E. 96 is not divisible by $9 \times 6 = 54$.

D. 72 is not divisible by $7 \times 2 = 14$.

C. 36 is divisible by $3 \times 6 = 18$.

Therefore 36 is the correct answer.

FOR INVESTIGATION

**4.1** Find all the two-digit positive integers that are divisible by the product of their digits.

**4.2** Find all the three-digit positive integers that are divisible by the product of their digits.

**4.3** The number 1352 has the property that it is a multiple of $13 \times 52$. Find the one other four-digit number '$pqrs$' which is a multiple of '$pq$' $\times$ '$rs$'.

**4.4** Find all the six-digit numbers '$pqrstu$' which have the property that '$pqrstu$' is a multiple of '$pqr$' $\times$ '$stu$'.

**5.** The record for travelling 100 m on a skateboard by a dog is 19.65 seconds. This was achieved by Jumpy in 2013. What was Jumpy's approximate average speed?

A 0.2 m/s B 0.5 m/s C 2 m/s D 2.5 m/s E 5 m/s

SOLUTION **E**

Jumpy's average speed was $\dfrac{100}{19.65}$ m/s. Now $\dfrac{100}{19.65}$ is approximately equal to $\dfrac{100}{20}$, which is equal to 5.

Therefore Jumpy's approximate average speed was 5 m/s.

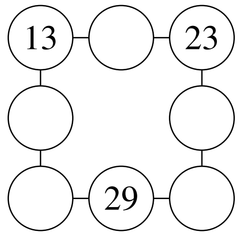
FOR INVESTIGATION

**5.1** According to *Guinness World Records* the fastest skateboard speed achieved in a standing position is 146.73 km/h by Peter Connolly, at Les Éboulements in Quebec, Canada, on 16 September 2017.

At this speed, to the nearest 5 seconds, how long would it take to cover 1 km?

**6.** When this prime number square is completed, the eight circles contain eight different primes, and each of the four sides has total 43.
What is the sum of the five missing primes?

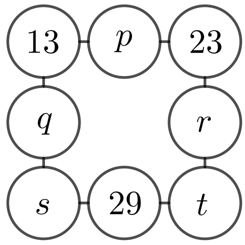
A 51  B 53  C 55  D 57  E 59




SOLUTION **D**

The question asks us only to find the *sum* of the five missing primes. So, in the context of the JMC, all we need do is find this sum on the assumption that it is possible to complete the square with primes in the way described. In Problem 6.1 you are asked to show that this is actually possible.

We let the different primes used to complete the square be $p$, $q$, $r$, $s$ and $t$, placed as shown in the diagram on the right.




The total of the primes on each of the four sides of the square is 43.

Therefore from the top side of the square

$$13 + p + 23 = 43. \quad (1)$$

From the left-hand hand side of the square,

$$13 + q + s = 43, \quad (2)$$

and from the right-hand side,

$$23 + r + t = 43. \quad (3)$$

By adding equations (1), (2) and (3), we obtain

$$(13 + p + 23) + (13 + q + s) + (23 + r + t) = 43 + 43 + 43.$$

This last equation may be rerrranged as

$$p + q + r + s + t + 13 + 23 + 13 + 23 = 43 + 43 + 43,$$

that is,

$$p + q + r + s + t + 72 = 129.$$

Therefore

$$\begin{aligned} p + q + r + s + t &= 129 - 72 \\ &= 57. \end{aligned}$$

FOR INVESTIGATION

**6.1** Find the values of the primes $p$, $q$, $r$, $s$ and $t$.

**6.2** Consider this question with 43 replaced by 73. What is the sum of the five missing primes in this case?

**6.3** Find another prime $n$, with $n > 73$, such that replacing 43 by $n$ in this question results in a prime number square that may be completed in just one way.

**7.** What is the difference between the largest two-digit multiple of 2 and the smallest three-digit multiple of 3?

A 5 B 4 C 3 D 2 E 1

Solution **B**

The largest two-digit multiple of 2 is 98.

The smallest three digit multiple of 3 is 102.

The difference between these two numbers is $102 - 98$ which is equal to 4.

For investigation

**7.1** What is the difference between the largest two-digit number that is a multiple of 7 and the smallest three-digit number that is a multiple of 9?

**7.2** What is the difference between the largest five-digit number that is a multiple of 2 and the smallest six-digit number that is a multiple of 3?

**7.3** Prove that, for each positive integer $n$, the difference between the largest $n$-digit number that is a multiple of 2 and the smallest $(n + 1)$-digit number that is a multiple of 3, is 4.

> It helps if you know that a test for whether a number is divisible by 3 is that the sum of its digits is a multiple of 3.
>
> For example, $2 + 0 + 2 + 3 = 7$. Since 7 is not a multiple of 3 we can deduce that 2023 is not a multiple of 3. On the other hand, $2 + 0 + 2 + 5 = 9$, and since 9 is a multiple of 3, we deduce that 2025 is a multiple of 3.
>
> You are asked to explain why this test works in Problem 7.4.

**7.4** Explain why a test for whether an integer is a multiple of 3 is that the sum of its digits is a multiple of 3.

**8.** How many of these six numbers are prime?

| | | | | | |
|---|---|---|---|---|---|
| $0^2 + 1^2$ | $1^2 + 2^2$ | $2^2 + 3^2$ | $3^2 + 4^2$ | $4^2 + 5^2$ | $5^2 + 6^2$ |

A 1 B 2 C 3 D 4 E 5

SOLUTION **D**

The only way to tackle this question is to calculate the values of the given numbers, and then decide which of them are prime numbers.

$0^2 + 1^2 = 0 + 1 = 1$, which is not prime.

$1^2 + 2^2 = 1 + 4 = 5$, which is prime.

$2^2 + 3^2 = 4 + 9 = 13$, which is prime.

$3^2 + 4^2 = 9 + 16 = 25$, which is not prime.

$4^2 + 5^2 = 16 + 25 = 41$, which is prime.

$5^2 + 6^2 = 25 + 36 = 61$, which is prime.

Therefore 4 of the given numbers are prime.

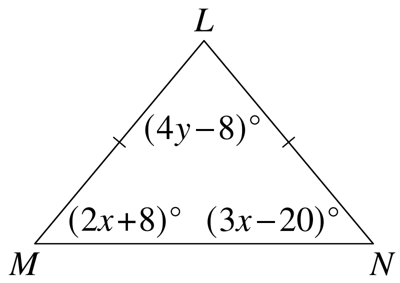
It is important to remember that 1 is considered *not* to be a prime number. There is no deep significance in this. It just turns out to be more convenient not to count 1 as a prime. For example the *Fundamental Theorem of Algebra* says that every positive integer greater than 1 has a unique factorization into primes. If 1 were regarded as a prime this theorem would need to be restated to refer to a unique factorization into primes greater than 1, as in that case, for example, both $2 \times 3 \times 3$ and $1 \times 2 \times 3 \times 3$ would count as prime factorizations of 18.

It is only in the last hundred years or so that the convention that 1 is not prime has become standard. There are still some books that give a definition of *prime number* that does not make it clear that 1 is not a prime number.

FOR INVESTIGATION

**8.1** Which is the smallest prime number of the form $n^2 + (n+1)^2$ that is greater than 61?

**9.** Triangle $LMN$ is isosceles with $LM = LN$.




What is the value of $y$?

A 15 B 17 C 19 D 21 E 23

SOLUTION **A**

Because $LM = LN$, it follows that $\angle LMN = \angle LNM$. Therefore $2x + 8 = 3x - 20$. Hence $x = 28$.

We therefore have $\angle LMN = (2 \times 28 + 8)^\circ = 64^\circ$ and $\angle LNM = (3 \times 28 - 20)^\circ = 64^\circ$.

Because the angles of a triangle add to $180^\circ$, we can now deduce that

$$\angle MLN = 180^\circ - 64^\circ - 64^\circ = 52^\circ.$$

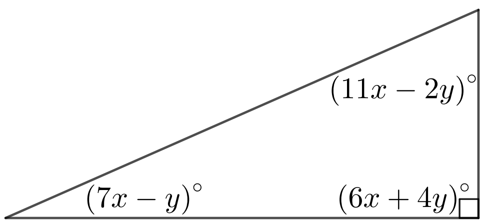
Therefore $4y - 8 = 52$. Hence $4y = 60$.

It follows that $y = 15$.

FOR INVESTIGATION

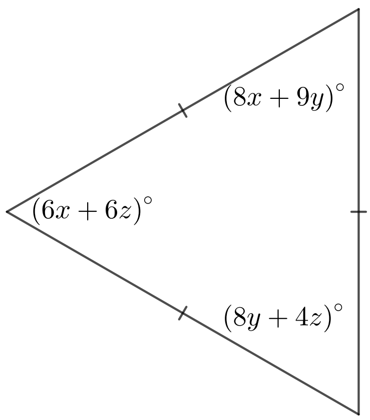
**9.1** The three angles of the right-angled triangle shown on the right are $(6x + 4y)^\circ$, $(7x - y)^\circ$ and $(11x - 2y)^\circ$.

Find the values of $x$ and $y$.



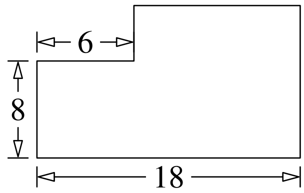

**9.2** The angles of the equilateral triangle shown on the right are $(8x + 9y)^\circ$, $(6x + 6z)^\circ$ and $(8y + 4z)^\circ$.

Find the values of $x$, $y$ and $z$.



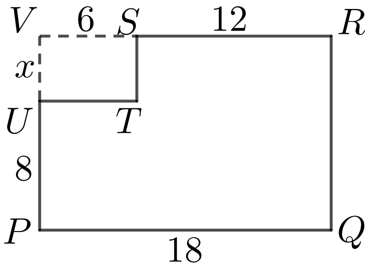
**10.** In the diagram, all distances shown are in cm. The perimeter of the shape is 60 cm. What is the area, in $\text{cm}^2$, of the shape?

A 192 B 204 C 212 D 232 E 252




Solution **A**

In the diagram on the right we have labelled the vertices of the shape, $P$, $Q$, $R$, $S$, $T$ and $U$ as shown.




We let $V$ be the point where the extension of $PU$ meets the extension of $RS$, and $x$ cm be the length of $UV$.

Since $STUV$ is a rectangle, the sum of the lengths of $UV$ and $VS$ is the same as the sum of the lengths of $ST$ and $TU$.

Therefore the sum of the lengths of $PV$ and $VR$ is equal to half the length of the perimeter of the shape.

Hence, $8 + x + 18 = 30$. Therefore $x = 30 - 8 - 18 = 4$. It follows thsat $PV$ has length 12 cm.

The area of the shape equals the area of the rectangle $RVPQ$ less the area of the rectangle $SVUT$.

Therefore the area of the shape is $(18 \times 12)\,\text{cm}^2 - (6 \times 4)\,\text{cm}^2 = 216\,\text{cm}^2 - 24\,\text{cm}^2 = 192\,\text{cm}^2$.

For investigation

**10.1** Suppose that the perimeter of the shape given in the question is $p$ cm with the lengths of edges as given. Find a formula, in terms of $p$, for the area, in $\text{cm}^2$, of the figure.

**11.** To save money, Scrooge is reusing tea bags. After a first 'decent' cup of tea, he dries the bag and uses two such dried bags to make a new 'decent' cup of tea. These bags are then dried again and four such re-dried bags make a 'decent' cup of tea. After that they are put on the compost heap.
How many 'decent' cups of tea can Scrooge get out of a new box of 120 tea bags?

A 480 B 240 C 210 D 195 E 180

Solution **C**

First, Scrooge uses the new tea bags to makes 120 'decent' cups of tea. He next uses the 120 dried tea bags to make $120 \div 2 = 60$ more 'decent' cups of tea. Finally he uses the 120 twice-dried tea bags to make a further $120 \div 4 = 30$ 'decent' cups of tea.

Therefore, in total, Scrooge makes $120 + 60 + 30 = 210$ 'decent' cups of tea.

For investigation

**11.1** How many 'decent' cups of tea could Scrooge get out of a new box of 150 tea bags?

**11.2** How many 'decent' cups of tea could Scrooge get out of a new box of $t$ tea bags?

**12.** One afternoon, Brian the snail went for a slither at a constant speed. By 1:50 pm he had slithered 150 centimetres. By 2:10 pm he had slithered 210 centimetres. When did Brian start his slither?

A Noon B 12:20 pm C 12:30 pm D 12:45 pm E 1 pm

SOLUTION **E**

In the 20 minutes from 1:50 pm to 2:10 pm Brian slithers $210 - 150$ centimetres, that is, 60 centimetres.

Therefore Brian's constant speed is $60 \div 20$ centimetres per minute, that is, 3 centimetres per minute.

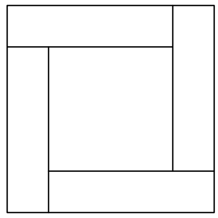
It follows that the number of minutes that Brian took to slither the first 150 centimetres was $150 \div 3$, that is, 50 minutes.

Hence Brian started his slither 50 minutes before 1:50 pm. So Brian started at 1 pm.

FOR INVESTIGATION

**12.1** At 2 pm Ermintrude starts shuffling along at a constant speed of 0.24 kilometres per hour. Find the digits $h$, $m$ and $n$ such that at the time $h$:$mn$ pm Ermintrude has shuffled '$hmn$' metres.

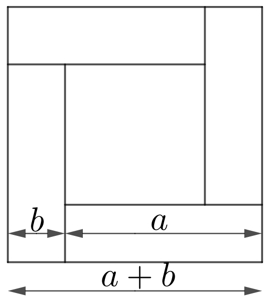
**13.** Four congruent rectangles are arranged as shown to form an inner square of area $20\,\text{cm}^2$ and an outer square of area $64\,\text{cm}^2$.
What is the perimeter of one of the four congruent rectangles?



A 6 cm B 8 cm C 9.75 cm D 16 cm E 20 cm

SOLUTION **D**

We let $a$ cm be the length of each rectangle and $b$ cm be the width of each rectangle.




It follows that the length of the sides of the outer square is $(a + b)$ cm.

Since the area of the outer square is $64\,\text{cm}^2$, it follows that $(a+b)^2 = 64$. Hence $a + b = 8$.

Each of the four rectangles has two sides of length $a$ cm and two sides of length $b$ cm. Therefore the perimeter of each rectangle is $2(a + b)$ cm.

Because $a + b = 8$, we deduce that the perimeter of one of the congruent rectangles is 16 cm.

FOR INVESTIGATION

**13.1** In this solution we found the answer without using the area of the inner square or finding the values of $a$ and $b$. Use the area of the inner square to find the values of $a$ and $b$.

**14.** In the addition shown, $x$ and $y$ represent different single digits. What is the value of $x + y$?

A 10 B 11 C 12 D 13 E 14

$$\begin{array}{r} 7\,7\,x \\ 6\,y\,x \\ +\ y\,y\,x \\ \hline 1\,x\,x\,7 \end{array}$$

SOLUTION **E**

From the units column we see that $3x$ has units digit 7. Since $x$ is a single digit, we deduce that $x = 9$. This gives $3x = 27$. Hence there is a carry of 2 from the units column to the tens column.

We can now see from the tens column that $7 + 2y + 2$ has units digit 9. Now $7 + 2y + 2 = 2y + 9$. Hence $2y$ has units digit 0. It follows that $y$ is 0 or 5.

If $y = 0$ the sum would be $779 + 609 + 9 = 1997$ which is not correct. We deduce that $y = 5$. In this case the sum becomes

$$\begin{array}{r} 7\,7\,9 \\ 6\,5\,9 \\ +\ \ 5\,5\,9 \\ \hline 1\,9\,9\,7 \end{array}$$

which is correct.

Since $x = 9$ and $y = 5$, we conclude that $x + y = 14$.

FOR INVESTIGATION

**14.1** In the addition sum

$$\begin{array}{r} x\,y\,x \\ z\,y\,x \\ +\ \ y\,z\,x \\ \hline w\,x\,y\,4 \end{array}$$

find the values of the digits $w$, $x$, $y$ and $z$.

**14.2** In the addition sum

$$\begin{array}{r} u\,u\,u \\ u\,u\,u \\ u\,u\,u \\ +\ w\,x\,y\,z \\ \hline 2\,0\,2\,3 \end{array}$$

$w$, $x$, $y$ and $z$ are odd digits.

Find the number '$wxyz$'.

**15.** My train was scheduled to leave at 17:48 and to arrive at my destination at 18:25. However, it started four minutes late, and the journey took twice as long as scheduled. When did I arrive?

A 19:39 B 19:06 C 19:02 D 18:29 E 17:52

SOLUTION **B**

There are 37 minutes from 17:48 to 18:25. Therefore, since the journey took twice as long as scheduled, it took 74 minutes, that is 1 hour and 14 minutes.

The train started 4 minutes late. Therefore it set off at 17:52.

It follows that I arrived 1 hour and 14 minutes after 17:52. So I arrived at 19:06.

FOR INVESTIGATION

**15.1** The next day the 17:48 train again set off late. The journey time was 50% longer than was scheduled. The train arrived at 18:58. At what time did it set off?

**16.** Amrita needs to select a new PIN. She decides it will be made up of four non-zero digits with the following properties:

i) The first two digits and the last two digits each make up a two-digit number which is a multiple of 11.
ii) The sum of all the digits is a multiple of 11.

How many different possibilities are there for Amrita's PIN?

A 1 B 2 C 4 D 8 E 16

SOLUTION **D**

A two-digit multiple of 11 has the form '$dd$', where $d$ is a non-zero digit. Therefore a PIN that has the first of Amrita's properties has the form '$aabb$', where $a$ and $b$ are non-zero digits.

The sum of the digits of '$aabb$' is $2(a + b)$. For this to be a multiple of 11, we need $a + b$ to be a multiple of 11. Because $a$ and $b$ are non-zero digits, $0 < a + b \leq 9 + 9 = 18$. Hence 11 is the only possible value of $a + b$.

Therefore the possibilities for Amrita's PIN are the numbers of the form '$aabb$', where $a$ and $b$ are non-zero digits with $a + b = 11$. Hence the possibilities for Amrita's PIN are

2299, 3388, 4477, 5566, 6655, 7744, 8833 and 9922.

It follows that there are 8 possibilities for Amrita's PIN.

FOR INVESTIGATION

**16.1** Amrita replaced her second condition by

ii) The sum of all the digits is a multiple of 3.

How many different possibilities are there now for Amrita's PIN?

**17.** Two numbers $p$ and $q$ are such that $0 < p < q < 1$.
Which is the largest of these expressions?

A $q - p$ B $p - q$ C $\frac{p+q}{2}$ D $\frac{p}{q}$ E $\frac{q}{p}$

Solution **E**

Since $0 < p < q < 1$,

$$q - p < q < 1, \qquad p - q < p < 1, \qquad \frac{p+q}{2} < \frac{1+1}{2} = 1,$$
$$\frac{p}{q} < 1 \qquad \text{and} \qquad \frac{q}{p} > 1.$$

Therefore $\frac{q}{p}$ has the largest value, as it is greater than 1 while all the other expressions have values which are less than 1.

For investigation

**17.1** The question does not tell us the values of $p$ and $q$, but only that they satisfy $0 < p < q < 1$. So, in the context of the JMC, it is safe to assume that the answer is independent of the actual values of $p$ and $q$. This means that you could answer the question by choosing particular values of $p$ and $q$ that satisfy the given inequalities.

This is not a sound mathematical method and would not gain you any marks if you were asked to give a mathematical justification for your answer. However, trying particular values for $p$ and $q$ is a good way to begin thinking about the question. It is also provides a way to check your answer.

Put $p = \frac{1}{4}$ and $q = \frac{3}{4}$, so that $0 < p < q < 1$.

For these values of $p$ and $q$ arrange the expressions

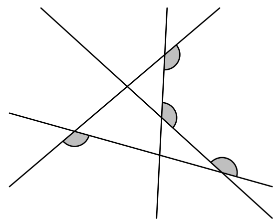
$$q - p, \quad p - q, \quad \frac{p+q}{2}, \quad \frac{p}{q}, \quad \frac{q}{p}$$

in order of their values from smallest to largest.

**17.2** Suppose that $2 < p < 3 < q < 4$. What can you deduce about the relative sizes of the expressions given in the question?

**18.** What is the sum of the four marked angles in the diagram?

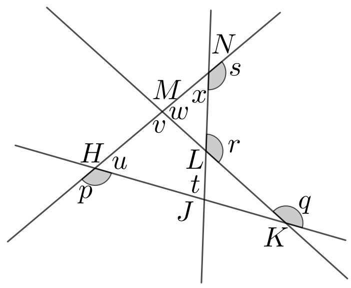
A $540°$ B $560°$ C $570°$ D $600°$ E $720°$



SOLUTION **A**

There are lots of ways to tackle this problem using basic properties of angles in a triangle and on a line. We make use of the *External Angle Theorem*. If you have not met this theorem before, see Problem 18.1. For an alternative method, see Problem 18.2.

We use $H$, $J$, $K$, $L$, $M$ and $N$ to label the points where the lines meet, and $p$, $q$, $r$, $s$, $t$, $u$, $v$, $w$ and $x$ for the sizes, in degrees, of the angles, as shown in the diagram on the right.




By the External Angle Theorem, we have

from the triangle $HJN$, $p = x + t$, (1)

from the triangle $HKM$, $q = u + v$, (2)

from the triangle $MLN$, $r = x + w$, (3)

and from the triangle $HJN$, $s = u + t$. (4)

By adding the equations (1), (2), (3) and (4), we deduce that

$$p + q + r + s = 2(x + t + u) + (v + w). \qquad (5)$$

Now $x + t + u$ is the sum, in degrees, of the angles of the triangle $HJN$ and is therefore equal to 180. Also $v + w$ is the sum, in degrees, of the angles on the line $HN$ at the point $M$, and so is equal to 180. It therefore follows from (5) that

$$p + q + r + s = 2 \times 180 + 180 = 540.$$

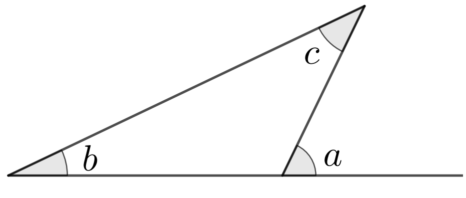
Hence the sum of the four marked angles in the diagram is $540°$.

FOR INVESTIGATION

**18.1** The *External Angle Theorem* says that the external angle of a triangle is the sum of the two opposite internal angles.

In terms of the diagram it says that $a = b + c$.

Prove the External Angle Theorem.




**18.2** Express $p$, $q$, $r$ and $s$ in terms of the interior angles of the quadrilateral $HKLN$. Then use the fact that the sum of the interior angles of $HKLN$ is $360°$ to find $p + q + r + s$.

**19.** In a football match, Rangers beat Rovers 5 – 4. The only time Rangers were ahead was after they scored the final goal. How many possible half-time scores were there?

A 9 B 10 C 15 D 16 E 25

Solution **D**

The final score was 5 – 4. Either this was also the score at half-time, or Rangers were not ahead at half-time. In the latter case, Rangers had scored at most 4 goals at half-time, and not more goals than Rovers.

Therefore, the possible half-time scores were, with the Rangers score given first:

5 – 4,

4 – 4,

3 – 3, 3 – 4,

2 – 2, 2 – 3, 2 – 4,

1 – 1, 1 – 2, 1 – 3, 1 – 4,

0 – 0, 0 – 1, 0 – 2, 0 – 3, 0 –4.

This makes a total of 16 possible half-time scores.

For investigation

**19.1** In another football match Rovers got their revenge by beating Rangers 6 – 5. The only time Rovers were ahead was after they scored the final goal.

How many possible half-times scores were there?

**19.2** Rangers had been playing Rovers at football twice a year for 20 years.

It turned out that over this period Rangers had scored 61 goals in these matches while Rovers had scored 60 goals. It was noticed that until they scored the final goal in the last match there was no time in the whole twenty years when Rangers had scored more goals than Rovers.

How many possibilities are there for how the numbers of goals scored were divided between the two teams after 10 years?

**20.** Each cell in the crossnumber is to be filled with a single digit.

| **Across** | **Down** |
|---|---|
| 1. A cube | 1. A prime |
| 2. A square | |

| | |
|---|---|
| 1 | |
| 2 | |

Which of these could be the sum of the four digits in the crossnumber?

A 17 B 16 C 15 D 14 E 13

Solution **A**

The only two-digit cubes are 27 and 64. Hence 1 Across is either 27 or 64.

We first consider the case when 1 Across is 27. It follows that 1 Down is a prime with tens digit 2. Hence 1 Down is either 23 or 29.

If 1 Down is 23, then 2 Across is a two-digit square with tens digit 3. Therefore 2 Across is 36.

If 1 Down is 29, then 2 Across would need to be a two-digit square with tens digit 9, but no such square exists.

Next, we consider the case when 1 Across is 64. It follows that 1 Down is a prime with tens digit 6. Hence 1 Down is either 61 or 67.

If 1 Down is 61, then 2 Across is a two-digit square with tens digit 1. Therefore 2 Across is 16.

If 1 Down is 27, then 2 Across would need to be a two-digit square with tens digit 7, but no such square exists.

Therefore the crossnumber may be completed in two ways. These are shown on the right.

| | |
|---|---|
| [1] 2 | 7 |
| [2] 3 | 6 |

| | |
|---|---|
| [1] 6 | 4 |
| [2] 1 | 6 |

The sums of the four digits in these crossnumbers are

$$2 + 7 + 3 + 6 = 18 \quad \text{and} \quad 6 + 4 + 1 + 6 = 17.$$

We therefore see that of the given options the only one which could be the sum of the digits in the crossnumber is 17.

For investigation

**20.1** Each cell in this crossnumber is to be filled with a single non-zero digit.

| **Across** | **Down** |
|---|---|
| 1. A prime | 1. A prime |
| 3. A prime | 2. A prime |

| | |
|---|---|
| 1 | 2 |
| 3 | |

(a) In how many different ways can this crossnumber be completed using four different digits?

(b) In how many different ways can this crossnumber be completed if the digits used do not need to be all different?

**21.** Eleanor's Elephant Emporium has four types of elephant. There are twice as many grey elephants as pigmy elephants, three times as many white elephants as grey elephants and four times as many pink elephants as white elephants. There are 20 more white elephants than pigmy elephants.
How many elephants are in Eleanor's Emporium?

A 123 B 132 C 213 D 231 E 312

SOLUTION **B**

We let $g$, $p$, $w$ and $q$ be the number of grey elephants, pigmy elephants, white elephants and pink elephants, respectively.

The information provided in the question gives us the following equations.

$$\begin{aligned} g &= 2p, && (1)\\ w &= 3g, && (2)\\ q &= 4w, && (3)\\ w &= p + 20. && (4) \end{aligned}$$

By (1) and (2), we have

$$w = 6p$$

and therefore, from (4),

$$6p = p + 20.$$

Hence

$$5p = 20$$

and therefore

$$p = 4.$$

It now follows from equations (1), (2) and (3) that $g = 8$, $w = 24$ and $q = 96$. Hence $g + p + w + q = 8 + 4 + 24 + 96 = 132$.

Therefore the number of elephants in Eleanor's Emporium is 132.

FOR INVESTIGATION

**21.1** Sheila's Sheep Shop sells four breeds of sheep.

There are half as many Herdwicks as Dorsets. The number of Dorsets is double the number of Suffolks. There are six times as many Suffolks as Cheviots.

There are 20 more Herdwicks than Cheviots.

What is the total number of sheep in Sheila's shop?

**22.** The positive integers from 1 to 9 inclusive are placed in the grid, one to a cell, so that the product of the three numbers in each row or column is as shown.
What number should be placed in the bottom right-hand cell?

A 9 B 6 C 4 D 3 E 2

| | | | |
|---|---|---|---|
| | | | 18 |
| | | | 105 |
| | | ? | 192 |
| 56 | 180 | 36 | |

Solution **B**

Of the six products of the numbers in the rows and the columns, 105 and 56 are the two that have 7 as a factor. Hence the integer 7 is placed in the cell in the middle row and the left-hand column.

Similarly, as 105 and 180 are the only products that have 5 as a factor, the integer 5 is placed in the middle row and middle column.

Since $105 = 7 \times 5 \times 3$, the integer 3 is placed in the middle row and right-hand column.

| | | | |
|---|---|---|---|
| | | | 18 |
| 7 | 5 | 3 | 105 |
| | | ? | 192 |
| 56 | 180 | 36 | |

The product of the three integers in the right-hand column is 36. Therefore the product of the two integers in this column other than 3 needs to be 12, because $36 = 3 \times 12$. These two integers cannot be 3 and 4, because the 3 is already in the right hand column. Therefore they are 2 and 6. So the number in the bottom right-hand cell is either 2 or 6.

Now if 2 were in the bottom right-hand cell, the product of the other two numbers in the bottom row would need to be 96, because $192 = 2 \times 96$. However the largest number that is the product of two of the integers from 1 to 9 is 72, that is, $8 \times 9$.

We can therefore conclude that the number that should be placed in the bottom right hand cell is 6.

In the context of the JMC we can take it for granted that the remaining numbers can be placed in the grid so that the row and column totals are as given. For a complete answer you would need to check that this is possible. You are asked to do this in Problem 21.1.

For investigation

**22.1** Show that it is possible to place the positive integers from 1 to 9, inclusive, in the grid so that the row and column products are as shown in the question above.

**22.2** In each of the cases shown on the right decide if it possible to place the positive integers from 1 to 9, inclusive, in the grid, one to a cell, so that the products of the three numbers in each row or column are as shown.

$(a)$

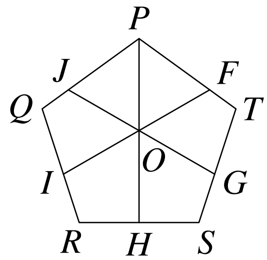
| | | | |
|---|---|---|---|
| | | | 56 |
| | | | 45 |
| | | | 144 |
| 80 | 189 | 24 | |

$(b)$

| | | | |
|---|---|---|---|
| | | | 72 |
| | | | 140 |
| | | | 36 |
| 48 | 168 | 90 | |

**23.** Regular pentagon $PQRST$ has centre $O$. Lines $PH$, $FI$ and $GJ$ go through $O$. The six angles at $O$ are equal.
What is the size of angle $TGO$?

A $60°$ B $72°$ C $75°$ D $76°$ E $78°$



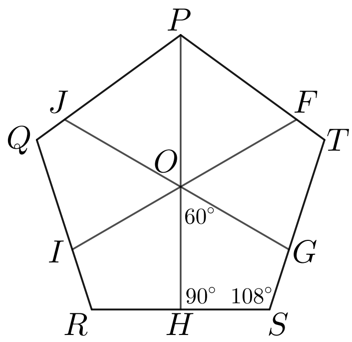

Solution **E**

We consider the angles in the quadrilateral $OHSG$.

Because the six angles at $O$ are equal, $\angle HOG = 360° \div 6 = 60°$.

$\angle HSG$ is the angle of a regular pentagon and is therefore $108°$.

By the symmetry of the figure, $\angle SHO = \angle RHO$. Therefore both of these angles are $90°$. [You are asked to make this more precise in Problem 23.1.]




The sum of the angles in a quadrilateral is $360°$. It follows that

$$\angle SGO = 360° - (60° + 108° + 90°) = 102°.$$

Because angles on a line have sum $180°$, we can now deduce that

$$\angle TGO = 180° - \angle SGO = 180° - 102° = 78°.$$

For investigation

**23.1** The solution above made the rather imprecise statement that $\angle SHO = \angle RHO$, "by the symmetry of the figure ".

(a) Prove that the triangles $QOP$ and $TOP$ are congruent.

(b) By considering the angles in the quadrilaterals $PTSH$ and $PQRH$ deduce that $\angle SHO = \angle RHO = 90°$.

**23.2** (a) Find the formula in terms of $n$ for the sum of the interior angles of a polygon with $n$ edges.

(b) Use your formula to deduce that the sum of the interior angles of a quadrilateral is $360°$.

(c) Use your formula to deduce that the interior angles of a regular pentagon are each $108°$.

**24.** Beatrix was born in this century. On her birthday this year, her age was equal to the sum of the digits of the year in which she was born. In which of these years will her age on her birthday be twice the sum of the digits of that year?

A 2027 B 2029 C 2031 D 2033 E 2035

Solution **E**

The most straightforward method is to try each of the options in turn. This is our first method. However, we can only use this method because we are given the options to choose between.

Our second method uses algebra. It enables us to answer the question without being given options to choose between.

Method 1

A. Suppose that in 2027 Beatrix's age will be twice the sum of the digits of 2027. Then her age will then be $2 \times (2 + 0 + 2 + 7) = 2 \times 11 = 22$.

It would then follow that Beatrix was born in the year $2027 - 22 = 2005$. Therefore on her birthday this year her age was $2023 - 2005 = 18$. However 18 is not the sum of the digits of her supposed birth year 2005. We conclude that A is not the correct option.

B. Suppose that in 2029 Beatrix's age will be twice the sum of the digits of 2029. Then her age will then be $2 \times (2 + 0 + 2 + 9) = 2 \times 13 = 26$.

It would then follow that Beatrix was born in the year $2029 - 26 = 2003$. Therefore on her birthday this year her age was $2023 - 2003 = 20$. However 20 is not the sum of the digits of her supposed birth year 2003. We conclude that B is nor the correct option.

C. Suppose that in 2031 Beatrix's age will be twice the sum of the digits of 2031. Then her age will then be $2 \times (2 + 0 + 3 + 1) = 2 \times 6 = 12$.

It would then follow that Beatrix was born in the year $2031 - 12 = 2019$. Therefore on her birthday this year her age was $2023 - 2019 = 4$. However 4 is not the sum of the digits of her supposed birth year 2019. We conclude that C is nor the correct option.

D. Suppose that in 2033 Beatrix's age will be twice the sum of the digits of 2033. Then her age will then be $2 \times (2 + 0 + 3 + 3) = 2 \times 8 = 16$.

It would then follow that Beatrix was born in the year $2033 - 16 = 2017$. Therefore on her birthday this year her age was $2023 - 2017 = 6$. However 6 is not the sum of the digits of her supposed birth year 2017. We conclude that D is nor the correct option.

Having eliminated options A, B, C and D, we can conclude, in the context of the Junior Mathematical Challenge, that the one remaining option is correct.

For a complete answer you would need to check that option E is correct. We leave this to the reader to check in Problem 24.1.

METHOD 2

Since Beatrix was born in this century, we may suppose that she was born in the year '$20xy$' where $x$ and $y$ are digits.

Because '$20xy$' represents the number $2000 + 10x + y$, this means that Beatrix's age on her birthday this year was $2023 - (2000 + 10x + y) = 23 - 10x - y$. Since her age was then equal to the sum of the digits of her birth year, we have

$$23 - 10x - y = 2 + 0 + x + y.$$

It follows that

$$11x + 2y = 21. \qquad (1)$$

Since $x$ and $y$ are non-negative integers, the only solution of (1) is $x = 1,\ y = 5$. [You are asked to check this in Problem 24.2.] Therefore Beatrix was born in 2015.

Now suppose Beatrix's age on her birthday in the year '$20wz$' will be twice the sum of the digits of that year. Her age in '$20wz$' will be $(2000 + 10w + z) - 2015 = 10w + z - 15$. Therefore,

$$10w + z - 15 = 2(2 + 0 + w + z).$$

It follows that

$$8w - z = 19. \qquad (2)$$

Since $w$ and $z$ are integers in the range from 0 to 9, the only solution of (2) is $x = 3,\ y = 5$. [You are asked to check this in Problem 24.3.]

Therefore the year in which Beatrix's age on her birthday will be twice the sum of the digits of that year is 2035.

FOR INVESTIGATION

**24.1** Check that option E is correct.

**24.2** Show that the only solution of the equation $11x+2y = 21$ in which $x$ and $y$ are non-negative integers is $x = 1,\ y = 5$.

**24.3** Show that the only solution of the equation $8w - z = 19$ in which $w$ and $z$ are integers in the range from 0 to 9 is $w = 3,\ z = 5$.

**24.4** Pat was born last century. On her birthday this year, her age was equal to three times the sum of the digits of the year in which she was born. In which year was Pat born?

**24.5** In the thirteenth century Mindaugas was crowned King of Lithuania. This year, the number of years since this coronation will be 70 times the sum of the digits of the year in which it took place.

In which year was Mindaugas crowned?

**25.** Granny gave away her entire collection of antique spoons to three people. Her daughter received 8 more than a third of the total; her son received 8 more than a third of what was then left; finally her neighbour received 8 more than a third of what was then left. What is the sum of the digits of the number of spoons which were in Granny's collection?

A 14 B 12 C 10 D 8 E 6

SOLUTION **B**

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METHOD 1

Suppose Granny began with $x$ spoons. She gives $\frac{1}{3}x + 8$ spoons to her daughter. Therefore the number of spoons she has left is $x - (\frac{1}{3}x + 8) = \frac{2}{3}x - 8$.

The same formula applies at the second stage, with $x$ replaced by $\frac{2}{3}x - 8$. So, after giving spoons to her son, Granny is left with $\frac{2}{3}(\frac{2}{3}x - 8) - 8$ spoons.

Similarly, the same formula applies at the third stage with $x$ replaced by $\frac{2}{3}(\frac{2}{3}x - 8) - 8$.

So, after giving spoons to her neighbour, Granny is left with $\frac{2}{3}(\frac{2}{3}(\frac{2}{3}x - 8) - 8) - 8$ spoons.

Now $\frac{2}{3}(\frac{2}{3}(\frac{2}{3}x - 8) - 8) - 8 = \frac{8}{27}x - \frac{8\times19}{9}$. [You are asked to check this in Problem 25.1.] Therefore, Granny gives away her entire collection of spoons provided that $\frac{8}{27}x - \frac{8\times19}{9} = 0$. This gives $\frac{8}{27}x = \frac{8\times19}{9}$. Hence

$$x = \frac{27}{8} \times \frac{8\times19}{9} = 57.$$

Therefore Granny began with 57 spoons.

The sum of the digits of 57 is 12.

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METHOD 2

In this method we simplify the algebra by working backwards.

We have seen in Method 1 that if Granny begins with $x$ spoons, after giving away 8 more than a third of her spoons, she is left with $\frac{2}{3}x - 8$ spoons.

We put $y = \frac{2}{3}x - 8$. This gives $x = \frac{3}{2}(y + 8)$. Therefore if Granny is left with $y$ spoons, she began with

$$\frac{3}{2}(y + 8) \qquad (1)$$

spoons.

Hence by putting $y = 0$ in the formula (1), we see that Granny had $\frac{3}{2}(0 + 8) = 12$ spoons before giving all her remaining spoons to her neighbour.

Similarly, by putting $y = 12$ in (1), we see that Granny had $\frac{3}{2}(12 + 8) = 30$ spoons before giving spoons to her son.

Finally, putting $y = 30$ in (1), we see that before giving spoons to her daughter, Granny had $\frac{3}{2}(30 + 8) = 57$ spoons. The sum of the digits of 57 is 12.

For investigation

**25.1** Check that in Method 1 the equation

$$\tfrac{2}{3}(\tfrac{2}{3}(\tfrac{2}{3}x - 8) - 8) - 8 = \tfrac{8}{27}x - \tfrac{8\times 19}{9}$$

is correct.

**25.2** Granny also had a collection of china cats. She gave 27 more than a quarter of her collection to her daughter. Her son was next given 27 more than a quarter of the cats that were then left. Finally, she gave her neighbour 27 more than a quarter of the cats that were then left.

Granny had then given away her entire collection of china cats. How many were in her collection to begin with?

**25.3** Granny also had a collection of teapots. She gave $k$ more than $\frac{1}{p}$th of her collection to her daughter. Her son was next given $k$ more than $\frac{1}{p}$th of the teapots that were then left. Finally, she gave her neighbour $k$ more than $\frac{1}{p}$th of the teapots that were then left.

Granny had then given away her entire collection of teapots.

(a) Find a formula in terms of $k$ and $p$ for the number of teapots that Granny began with.

(b) Substitute $k = 8$ and $p = 3$ into your formula to check that it gives the correct answer of 57 to Question 25.

(c) Use your formula to check your answer to Problem 25.2.